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The irreversibility of the equations of classical dynamics (the Hamilton equations and the Liouville equation) in the space with multifractal time is demonstrated. The time is given on multifractal sets with fractional dimensions. The last depends on densities of Lagrangians in a given time moment and in a given point of space. After transition to sets of time points with the integer dimension the obtained equations transfer in the known equations of classical dynamics. Production of an entropy is not equally to zero in space with multifractal time, i.e. the classical systems in this space are non-closed.

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I. INTRODUCTION

The dynamic equations of the physical theories are reversible, it is well known. The kinetic equations of the statistical theory are irreversible. The irreversibility the Boltzmann's statistical theory was in due time the main reason of non-recognition by Poincare of Boltzmann's statistical theory. For the irreversibility introducing in the physical dynamical equations (for example, in the equation of the Liouville) it is necessary to introduce the dissipation terms [1], [2] or the functionals of a microscopic entropy and time [3] ensuring realization of the second law of thermodynamics. In the first case irreversibility in the dynamic equations arises as a sequence of mathematical approaches. Prigogin's the point of view is consists in recognizing the primacy of irreversible processes and it seems intuitively more reasonable. Are more general, than mentioned, a methods of introducing of the irreversibility in the dynamical equations of physics exist? Is it possible the reversibility of the equations of the dynamical theories to introduce as result of approximate transition from more rigorous the dynamical irreversible equations (these dynamic equations may be obtained as a result of generalization of the known equations) to the idealized and reversible, but approximate equations? The purpose of a note is to introduce one of a possible generalizations of the dynamical theories of physics realized by replacement of time with topological dimension equal unity on "multifractal" time (for the first time it was introduced in [4]). In the mentioned theory the time is characterized in each time and space points by fractional (fractal) dimensions (FD)$d_t(\mathbf{r}(t),t)$. The marked replacement dimension of time by fractional dimensions gives in the origin of irreversibility in the dynamical equations of physics and the existence of irreversibility in our world may be interpreted by new reason. It is not contradicts (for FD is small differs from unity for weak physical fields on the Earth) an experimental data, and allows to receive a new interesting physical results. The equations of the classical mechanics (the equations of the Hamilton - Liouville) are chosen for research as an typical example of dynamic systems.

II. EQUATION OF A MECHANICS WITH TIME DEFINED ON MULTIFRACTAL SETS

The method of generalization of the classical mechanics equations is founded on the new model of an approach to the problem of a nature of time [4].This model is consists in replacement of usual time by the time defined on a multifractal subsets s_t of continuously set M_t (the measure carrier). The multifractal set S_t consists of subsets s_t, i.e. very small time intervals (in further named "points"), which are also multifractal, and each of them is characterized in turn by its global fractal dimension FD) $d_t(\mathbf{r}(t),t)$ (defined as box-dimension, [5], [6] and so on) that depends of a nature of sets s_t, and depends at coordinates and time (see [4], [7]). So each time subset is characterized by its global fractal dimension $d_t(\mathbf{r}(t),t)$ which characterize the scaling characteristics for this subset. The continuity $d_t(\mathbf{r}(t),t)$ is supposed. The new approach to a nature of time is consists in the replacing the usual time points of time axe by selection for describing of the time's intervals (the "points" on a time axis consisting of sets s_t that defined on the measure carrier M_t) only the "points" that characterized by sets s_t. The time axe (or, in other selections time plane or time volume R_n) is the carrier of measure of all the multifractal time subsets s_t defined on it. The researching of the dynamic equations and physical quantities (in particular, the entropy) with time "points" with fractal characteristics defined on multifractal sets s_t with FD $d_t(\mathbf{r}(t),t)$, lead to irreversibility of all dynamic theories used in physics. It is stipulated by an openness of dynamic systems with multifractal time (role of a thermostat plays set R_n) that is appears in a time dependencies of all physical and mathematical (except for zero) objects. For describing of small changes of functions defined on multifractal time sets, it is impossible to apply ordinary or fractional (in sense of the Riemann - Liouville) derivatives and integrals, since to different time points there corresponds to different fractal dimensions. For describing of changes of such functions need's introduction of generalized fractional derivatives and integrals [4]. In this note the multifractal properties of the space sets $s_{\mathbf{r}}$ are not considered, since the irreversibility of the dynamic physical equations arises already at the using only of multifractal time (see also [7])and global FD of small time sets intervals s_t.

III. GENERALIZED FRACTIONAL DERIVATIVES AND INTEGRALS ON MULTIFRACTAL SET S_T OF TIME POINTS

It is necessary if we want to describe the dynamics of time-dependent functions determined on multifractal set S_t to enter the functionals that extends the fractional derivatives and integrals of the Riemann - Liouville on the set S_t with FD $d_t(\mathbf{r}(t),t))$ that is different in each subsets s_t [4])

$$D^{d_t}_{+,t} f(t) = \left(\frac{d}{dt}\right)^n \int\limits_a^t dt' \frac{f(t')}{\Gamma(n-d_t(t'))(t-t')^{d_t(t')-n+1}} \tag{1}$$

$$D^{d_t}_{-,t} f(t) = (-1)^n \times \\ \times \left(\frac{d}{dt}\right)^n \int\limits_t^b dt' \frac{f(t')}{\Gamma(n-d_t(t'))(t'-t)^{d_t(t')-n+1}} \tag{2}$$

where Γ-is Euler gamma -function,$a < b$, a and b is stationary values selected on an axis (from $-\infty$ to ∞), $n-1 \le d_t < n$, $n = \{d_t\}+1$, $\{d_t\}$ is an integer part of $d_t \ge 0$, $n = 0$ for $d_t < 0$, $d_t = d_t(\mathbf{r}(t),t)$-is the fractal dimensions (FD). The dependencies FD from time and space coordinates are defined by the Lagrangians's densities of a viewed problem [4], [7], [8]. The generalized fractional derivative (GFD) (1)-(2) coincide with fractional derivatives or fractional integrals of the Riemann - Liouville [9] in the case $d_t = const$. At $d_t = n+\varepsilon(t)$, $\varepsilon \to 0$ GFD are represented by usual derivatives and integrals [4]. The functions and integrals in (1)-(2) are considered as generalized functions given on the set of finitary functions [10]. The definitions the GFD (1)-(2) allow to describe the dynamics of functions defined on multifractal sets and GFD substitute (for such functions) the usual or fractional differentiation and integration (GFD partially conserve the memory about of the last time events).

IV. HAMILTON EQUATIONS

The Hamilton equations for system from N of classical particles with identical masses m on the set with multifractal time (i.e. time defined on multifractal set S_t) reads:

$$D^{d_t}_{+,t}\mathbf{r}_i = \frac{\partial H}{\partial \mathbf{p}_i}, \quad D^{d_t}_{-,t}\mathbf{p}_i = -\frac{\partial H}{\partial \mathbf{r}_i}, \quad \mathbf{p}_i = D^{d_t}_{+,t}\mathbf{r}_i \tag{3}$$

$$H = \sum_{i=1..N} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2}\sum_{i\neq j=1..N} V(|\mathbf{r}_i - \mathbf{r}_j|) \tag{4}$$

The equations (refeq3)-(ref4) differ from the classical Hamilton equations by replacement the derivatives with respect to time by GFD (refeq1)-(ref2) and coincide with the classical equations of a mechanics at $d_t = 1$. The equations for arbitrary function B dynamic variable $\mathbf{p},\mathbf{r}$ will look like

$$D^{d_t}_{+,t} B = \tilde{D}^{d_t}_{+,t} B + \frac{\partial H}{\partial \mathbf{p}_i}\frac{\partial B}{\partial \mathbf{r}_i} - \frac{\partial H}{\partial \mathbf{r}_i}\frac{\partial B}{\partial \mathbf{p}_i} \tag{5}$$

The figure $\tilde{D}^{d_t}_{+,t}$ in (5) differs from $D^{d_t}_{+,t}$ in (1) by replacement the complete derivative with respect to time t on a partial differential with respect to time t. Let's show, that the Hamiltonian function H in space with multifractal time is not integral of a motion of the equation (5), i.e. does not convert a right part of (5) in zero. Substitution H in (5) gives in

$$D^{d_t}_{+,t} H = \tilde{D}^{d_t}_{+,t} H \tag{6}$$

From the (1) follows, that equation (6) is of the form (for $d_t(\mathbf{r}(t),t) = 1+\varepsilon(\mathbf{r}(t),t)$, $\varepsilon \to 0$, when a simplifying assumption about lack at d_t, ε of explicit dependence from t is valid)

$$D^{d_t}_{+,t} H = \tilde{D}^{d_t}_{+,t} H = \frac{\varepsilon H}{\Gamma(1+\varepsilon)t^{d_t}} \tag{7}$$

and is equal to zero when $d_t(\mathbf{r}(t),t) = 1$. So, in space with multifractal time, at classical system with a Hamiltonian H that not depends explicitly at time (conservative systems) the GFD with respect to a total energy depends on time and decreases with the increases of time, i.e. in the model of multifractal time the rigorously conservative classical systems does not exist. For differs $d_t(\mathbf{r}(t),t)$ from unity by a little bit (that it is valid about it represents experimental data about time and results of [4]) the changing of energy of system will be very small. Let's consider the problem of change H with change of time in more general case ($\varepsilon = \varepsilon(\mathbf{r}(t),t)$). For this purpose we shall be restricted to a case, when FD of time $d_t(\mathbf{r}(t),t) = 1$ is not considerably differs from unity: $d_t = 1+\varepsilon(\mathbf{r}(t),t) = 1$, $|\varepsilon| \ll 1$. In this case GFD is represented as [4] (integral is calculated as the total of a principal value and residue in a singular point)

$$\tilde{D}^{1+\varepsilon}_{+,t} H \approx \frac{\partial}{\partial t} H \mp \frac{1}{2}\frac{\partial}{\partial t}[\frac{\varepsilon(\mathbf{r}(t),t) H}{\Gamma(1+\varepsilon(\mathbf{r}(t),t))}] + \\ + \frac{\varepsilon H}{\Gamma(1+\varepsilon)t^{d_t}}, \quad \varepsilon > 0, \quad d_t < 1 \tag{8}$$

$$\tilde{D}^{1+\varepsilon}_{+,t}H \approx \frac{\partial}{\partial t}[\frac{1}{\Gamma(1-\varepsilon)}H] \pm \frac{1}{2}\frac{\partial}{\partial t}[\frac{\varepsilon(\mathbf{r}(t),t)H}{\Gamma(1-\varepsilon)}] + + \frac{\varepsilon H}{\Gamma(1+\varepsilon)t^{d_t}}, \quad \varepsilon > 0, \quad d_t > 1 \qquad (9)$$

The selection of signs (plus or minus) in (8)-(9) is determined by sign of ε and requirements of a regularization of integrals and selection of FD d_t (greater or smaller unity). Let $d_t(\mathbf{r}(t),t) < 1$. In this case from (8) follows (for H do not containing explicit time dependence)

$$D^{d_t}_{+,t}H = \tilde{D}^{d_t}_{+,t}H \approx \pm\frac{1}{2}H\frac{\partial\varepsilon(\mathbf{r}(t),t)}{\partial t} + \frac{\varepsilon H}{\Gamma(1+\varepsilon)t^{d_t}} \qquad (10)$$

For $t \to \infty$, the basic contribution in the (10) imports corrections proportional to velocity of change FD $d_t(\mathbf{r}(t),t)$. The total energy conservative (in sense, that the Hamiltonian has not an explicit dependence at time) systems now in space with multifractal time is not conservative systems. It changes can be at $t \to \infty$ of any sign, and depend on a sign of derivatives with respect to time from the fractional correction to dimension of time ε. For $\varepsilon = 0$ the total energy of system is conserves and all relations coincide with known relations following from the dynamic equations of classical systems mechanics.

V. LIOUVILLE EQUATION

The equation of the Liouville for a N-partial distribution function $\rho(X,t)$ (X- are coordinate and impulses of particles of a $6N$-dimension phase space) is equivalent to the Hamilton equations for system from N of classical particles and is invariant in relation to transformations

$$\mathbf{r}_i \to \mathbf{r}_i, \quad \mathbf{p}_i \to -\mathbf{p}_i, \quad t \to -t, \quad i = 1,2,3,..N \qquad (11)$$

(equation is reversible). On set S_t of multifractal time complete derivative $\rho(X,t)$ will reads:

$$D^{d_t}_{+,t}\rho(X,t) = \tilde{D}^{d_t}_{+,t}\rho(X,t) + \frac{\partial H}{\partial \mathbf{p}_i}\frac{\partial\rho(X,t)}{\partial \mathbf{r}_i} - -\frac{\partial H}{\partial \mathbf{r}_i}\frac{\partial\rho(X,t)}{\partial \mathbf{p}_i} = \tilde{D}^{d_t}_{+,t}\rho(X,t) - L\rho(X,t) \qquad (12)$$

$$\mathbf{p}_i = D^{d_t}_{+,t}\mathbf{r}_i$$

where $D^{d_t}_{+,t}\rho$ is defined by (1), L-functional of the Liouville and differs from the functional of the known equation of the Liouville by replacement of ordinary derivatives with respect to time on GFD. For a demonstration of the irreversibility of expression (12) to transformations (11) we shall mark the following: the distribution functions $\rho(X,t)$ and $\rho(X_0,t_0)$ viewed in different moments t_0 and t are connected by the relation

$$\rho(X,t)dX = \rho(X_0,t_0)dX_0 \qquad (13)$$

in which because of change of fractal dimension with a change time $dX \neq dX_0$. Therefore $\rho(X,t) \neq \rho(X_0,t_0)$ and $\rho(X,t)$ evolves with a time. Complete derivative $D^{d_t}_{+,t}\rho(X,t)$, in particular and in that connection, is not equal to zero. Intuitively it is clear, that derivative $D^{d_t}_{+,t}\rho(X,t)$ is determined by a functional from function $\rho(X,t)$ equal to zero at $d_t = 1+\varepsilon = 1$. Let's designate this functional describing change $\rho(X,t)$ owing to interaction with "thermostat", by $\varphi = \varphi_0(\rho,d_t,t)\varepsilon(\mathbf{r}(t),t)$. The role of the thermostat plays the set M_t (being the carrier of a measure of a subsets of time points s_t and belonging to one of spaces R^n), as was already marked. The appealing of the functional φ is caused not to interior processes happening with change of energy inside system, but is determined by different properties of time sets s_t in different instant of time (change of dimension of s_t with a time changes). Complete derivative in this case will be equal to a functional φ and (12) will reads as the equation

$$\tilde{D}^{d_t}_{+,t}\rho(X,t) + \frac{\partial H}{\partial \mathbf{p}_i}\frac{\partial\rho(X,t)}{\partial \mathbf{r}_i} - \frac{\partial H}{\partial \mathbf{r}_i}\frac{\partial\rho(X,t)}{\partial \mathbf{p}_i} = = \varphi_0(\rho,d_t,t)\varepsilon(\mathbf{r}(t),t) \qquad (14)$$

The equation (14) is analogous of the Liouville equation of the classical systems with the time defined on multifractal sets. The analog of a collision integral in a right member (14) is stipulated by interaction with the carrier of a measure of multifractal set M_t and is equal's to zero if sets of time s_t is substitutes by sets with topological dimension equal to unity.

VI. PRODUCTION OF AN ENTROPY

Let's consider a classical system which is be found in an equilibrium state at the usual describing of the time.The production of an entropy in such system is equal to zero. Let's consider the production of the entropy $S = \int \rho(X,t)\ell n\rho(X,t)dX$ of same classical system defined on multifractal set of time points S_t (for $d_t = 1-\varepsilon < 1$):

$$D^{d_t}_{+,t}S = \int D^{d_t}_{+,t}[\rho\ell n\rho]dX \qquad (15)$$

Permissible, as well as earlier, that $d_t = 1+\varepsilon(\mathbf{r}(t),t)$, $|\varepsilon \ll 1|$. As $\rho(X,t)$ has for the equilibrium system at $d_t \neq 1$ the complete GFD which is non-equal zero, the right member (15) is not equal to zero. It means, that equilibrium systems does not exist in space with multifractal time. Really, as

$$D^{d_t}_{+,t}S = \int D^{d_t}_{+,t}[\rho\ell n\rho]dX \approx \frac{\varepsilon S}{t^{d_t}} \pm \frac{1}{2}\frac{\partial}{\partial t}(\varepsilon S) + \frac{\partial S}{\partial t} \neq 0 \qquad (16)$$

that (16) is an inequality and in a case $\frac{\partial S}{\partial t} = 0$. For $\frac{\partial\varepsilon}{\partial t} = 0$ the production of the entropy is positive. For

$\frac{\partial \varepsilon}{\partial t} \neq 0$, $\frac{\partial S}{\partial t} = 0$ the production of the entropy can have any sign (in particular, the entropy can be decreasing too). Let's mark in that circumstance, that all new results for behavior of the entropy are stipulated by the multifractality of the time and disappear after transition to time with topological dimension equal to unity.

VII. ABOUT CONNECTION FD D_T WITH LAGRANGIANS OF PHYSICAL FIELDS

In the monograph [4] the following approximating connection of fractional dimension of time $d_t(\mathbf{r}(t), t)$ with a Lagrangian density L of all physical fields in a point $\mathbf{r}(t)$ in an instant t (see also [7], [8]) is obtained:

$$d_t(\mathbf{r}(t), t) = 1 + \sum_i \beta_i L_i(\mathbf{r}(t), t) \tag{17}$$

where β_i is dimensional numerical factors ensuring a zero dimension of products $\beta_i L_i$. In [4] it is shown, that for coinciding the results founded on the (17) with results of the theory general relativity (GR) it is necessary (for gravitational forces) to choose $\beta = \frac{2}{c^2}$ (c- speed of light). For correspondence with results of a quantum mechanics, for electric fields the β_e has an order of magnitude $\beta_e = (2mc^2)^{-1}$ (m -is mass of particle or body creating the electric charge). The small differences FD from unity is satisfied by condition

$$\sum_i \beta_i L_i = \varepsilon \ll 1 \tag{18}$$

The connection ε with density of Lagrangians adds physical sense GFD (more in detail about it see [7]) and renders concrete relations obtained in the previous paragraphs.

VIII. CONCLUSIONS

The present note is devoted to the appendix of idea of the multifractal time offered in [4] [8], for researching of the problem of an irreversibility in large classical systems consisting from an identical objects. The following results, leading from this paper, on our sight, are essential:
1. The equations describing behavior of conservative systems (in usual time), are irreversible in space with multifractal time;
2. The neglecting by the fractionality of dimension of time and transition in space with topological dimension of time equal to unity allows to receive the known reversible equations of classical dynamics;
3. In space with the multifractal time there are no invariable objects, since the GFD with respect to stationary values are not equal to zero. From the physical point of view it is the reflection of non-stationary of the Universe with multifractal time. Last statement corresponds to mathematical exposition of behavior of physical objects and not contradict the exposition of the Einstein type of Universe , in which, in connection with its expansion, there are no invariable objects;
4. The quantity of the fractional additional to topological dimension of time member ε is determined by physical fields and depends on the density of energy that presents in the given moment in the given point of space . At the small densities of energy the corrections are very small. So, for the gravitational fields at distances more larger that gravitational radius (for example, for FD created by mass of the Earth on a surface of Earth) and for electric fields on atomic distances the value of ε is equal $\varepsilon \sim 10^{-8}$. Therefore the multifractal nature of time ($d_t \sim 1+\varepsilon$) does not contradicts an existing experimental data.

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